# PHY489 Week 5:
# Transition Probabilities

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FERMI'S "Golden Rule" for transition probabilities:

- measure of the probability $P(|i\rangle \to |f\rangle)$ in an interaction.

$$W = \frac{2\pi}{\hbar} |A|^2 \rho_f(E)$$

Matrix element $|\langle i|S|f\rangle|^2$

density of final States of energy $E$.

No further explaination given.

Interaction picture: $H = H_0 + H_{int}$

$H_0$: uncoupled Hamiltonian

$H_{int}$: coupled interaction term

with $H_{int} \ll H_0$ (perturbation theory) and gives rise to Wick's expansion / Dyson series.

Conventional normalization : $\psi = \frac{1}{\sqrt{N}} e^{-i(\vec{k}\cdot\vec{x} - \omega t)}$ ;

$$\int_{\mathbb{R}^3} d^3r \; \psi^* \psi = 1 .$$

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[FROM 2403 Notes Cause Miriam can't teach]

- Put the system in box of volume $V = L^3$ and enable interaction for a short time $T$.

  States are then normalized to $\langle \vec{k} | \vec{k}' \rangle = \delta_{\vec{k}\vec{k}'}$ .

  Dividing this by $T$ gives the transistion probability per unit time, then we can take the $T, V \to \infty$ limit.

- Periodic boundary conditions, so allowed momenta are

$$k_x = \frac{2\pi n_x}{L} , \quad k_y = \frac{2\pi n_y}{L} , \quad k_z = \frac{2\pi n_z}{L} .$$

Then, since the $k_i$'s are discrete, we sum:

$$\phi(x) = \sum_{\vec{k}} \left[ \frac{a_k e^{-ik\cdot x}}{\sqrt{2\omega_k}\sqrt{V}} + \frac{a_k^\dagger e^{ik\cdot x}}{\sqrt{2\omega_k}\sqrt{V}} \right]$$

as the field expansion.

Hence (in the nonrelativistic normalization) every time a field acts to create/annihilate a particle, it brings out an additional factor of $\dfrac{e^{\pm i k\cdot x}}{\sqrt{2\omega_k}\sqrt{V}}$.

For $i$ initial and $f$ final particles over the volume $V$ and time $T$, we have

$$= (2\pi)^4 \cdot \underbrace{\frac{1}{(2\pi)^4}\int_{-T/2}^{T/2} dt \int_V d^3\vec{x}\; e^{i(P_F - P_I)\cdot x}}_{\xrightarrow[V,T\to\infty]{} \delta^{(4)}(P_F - P_I)} \cdot \prod_f \frac{1}{\sqrt{2\omega_k}\sqrt{V}} \cdot \prod_i \frac{1}{\sqrt{2\omega_k}\sqrt{V}}$$

Therefore

$$\langle f | S - 1 | i \rangle_{VT} = i\mathcal{A}_{fi}^{VT} (2\pi)^4 \delta^{(4)}(P_F - P_I) \prod_f \frac{1}{\sqrt{2\omega_k}\sqrt{V}} \cdot \prod_i \frac{1}{\sqrt{2\omega_k}\sqrt{V}}$$

Since we can't measure individual momenta values (they don't resolve into a single state), it is only possible to measure all states about some finite region $\Delta\vec{k}$ in momentum space.

It is clear that in a region $\Delta k_x \Delta k_y \Delta k_z$, there are

$$\frac{L}{2\pi}\Delta k_x \frac{L}{2\pi}\Delta k_y \frac{L}{2\pi}\Delta k_z = \frac{V}{(2\pi)^3}\Delta k_x \Delta k_y \Delta k_z$$

States.

Taking $\Delta k_x \Delta k_y \Delta k_z \to d^3p$ and having $N$ particles in the final state, there will be

$$\prod_{f=1}^{N} \frac{V}{(2\pi)^3} d^3 p_f \qquad \left| \prod_f \frac{1}{\sqrt{2\omega_k}\sqrt{V}} \right|^2 \cdot \frac{V}{(2\pi)^3} d^3 p_f$$

States, which must be summed over by $p$.

Thus the transition probability (is now sensible because all quantities are finite) per unit time is

$$\frac{\omega_{VT}}{T} = \frac{1}{T} |A_{fi}^{VT}|^2 (2\pi)^8 \, |\delta_{VT}^{(4)}(P_F - P_i)|^2 \cdot \prod_f \frac{d^3 p_f}{(2\pi)^3 \cdot 2\omega_p} \prod_i \frac{1}{2\omega_i V} .$$

The $\delta_{VT}^{(4)}$ function: is it a $\delta$-function?

int. over $d^4p$ gives $\delta$-function

$$\int d^4p \, |\delta_{VT}^{(4)}(\vec{p})|^2 = \frac{1}{(2\pi)^8} \int d^4 p \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt' \int_V d^3\vec{x} \int_V d^3\vec{x}' \, e^{ip\cdot x - ip\cdot x'}$$

$$= \frac{1}{(2\pi)^8} (2\pi)^4 \delta^{(4)}(x-x') \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt' \int_V d^3\vec{x} \int_V d^3\vec{x}' \, \underset{\downarrow\,1}{e^{ip\cdot x - ip\cdot x'}}$$

$$= \frac{1}{(2\pi)^4} \int_{-T/2}^{T/2} dt \int_V d^3\vec{x}$$

$$= \frac{VT}{(2\pi)^4}$$

diverges as $V,T \to \infty$:

$$\lim_{V,T\to\infty} \left|(2\pi)^4 \delta^{(4)}_{VT}(p)\right|^2 = VT\,(2\pi)^4 \delta^{(4)}(p)$$

Hence $$\frac{\omega}{T} = |A_{fi}|^2 \cdot V(2\pi)\,\delta^{(4)}_{VT}(p_F - p_I) \cdot \prod_{\substack{\text{final}\\ \text{states}\\ f}} \frac{d^3p_f}{(2\pi)^3\, 2\omega_f} \cdot \prod_{\substack{\text{initial}\\ \text{states}\\ i}} \frac{1}{2\omega_i V}$$

Examine the Stat. factor $S$ later...

$$\boxed{\frac{\omega}{T} = |A_{fi}|^2\, V D \prod_{\substack{\text{initial}\\ \text{states}\\ i}} \frac{S}{2\omega_i V} \qquad \text{where} \quad D = (2\pi)^4 \delta^{(4)}(p_F - p_I) \prod_{\substack{\text{final}\\ \text{states}\\ f}} \frac{d^3p_f}{(2\pi)^3\, 2\omega_f}}$$

$\delta^{(4)}(p_F - p_I)$ ensures that E/momentum is conserved.

FACTORS OF V MUST CANCEL IN ORDER FOR THE $V,T \to \infty$ TO BE SENSIBLE.

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**Golden Rule for decays:** $1 \to 2+3+4+\ldots+N$.

Transition probability is $\frac{\omega}{T} = |A_{fi}|^2 \frac{1}{2\omega_1} D$. $\omega_1 \equiv E$.

✓ Factors of V have cancelled.

In particle's rest frame, $E_1 = \omega_1 \to m_1$, and the differential decay probability per unit time is

$$d\Gamma = \frac{1}{2m_1} |A_{fi}|^2 D.$$

The total decay is therefore $$\Gamma = \frac{1}{2m_1} \int_{\text{ALL FINAL STATES}} |A_{fi}|^2 D.$$

Observe that the decay with the largest energy release will proceed most rapidly (? - Diamond's notes)

$\longrightarrow$ A statistical factor $S$ arises from combinatorics of states (in $|A_{fi}|^2$).

## Golden Rule for Scattering:

$\frac{d\sigma}{d\Omega} \equiv$ differential cross-section.

$$\Rightarrow \quad d\sigma = \frac{\text{differential probability}}{\text{unit time} \times \text{unit flux}}$$

$$= \frac{A_{fi}^2}{4E_1E_2V} D \cdot \frac{1}{\text{flux}}$$

$$\text{Flux} = \frac{|\vec{v}_2 - \vec{v}_1|}{V}$$

$$= \frac{A_{fi}^2}{4E_1E_2} \frac{1}{|\vec{v}_1 - \vec{v}_2|} D$$

where $\vec{v}_1, \vec{v}_2$ are the 3-velocities of the colliding particles.

Thus, for $mn \to 1 + 2 + \ldots + N$ scattering,

$$d\sigma = \frac{S}{4E_mE_n} \frac{1}{|\vec{v}_1 - \vec{v}_2|} |A_{fi}|^2 D$$

Look at $S$ — next page.

$$\Rightarrow \quad \sigma = \frac{S}{4E_mE_n} \frac{1}{|\vec{v}_1 - \vec{v}_2|} \int_{\text{ALL FINAL STATES}} |A_{fi}|^2 \cdot (2\pi)^4 \delta^{(4)}(P_F - P_I) \cdot \prod_{k=1}^{N} \frac{d^3\vec{p}_k}{(2\pi)^3 \cdot 2E_k}$$

But where is the solid angle element $d\Omega$ in the integration? Arises from $D$ and the number of final states.

$D$ for 2-body states:

$$D = \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \cdot \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - \underbrace{p_I}_{=0 \text{ in CM frame!}})$$

$$= \frac{d^3p_1\, d^3p_2}{(2\pi)^6 \cdot 2^2 E_1 E_2} (2\pi)^3 \delta^{(3)}(\vec{p}_1 + \vec{p}_2)\,(2\pi)\,\delta(E_1 + E_2 - E_{TOT})$$

$\vec{p}_1 = -\vec{p}_2$ implicit

$$= \frac{d^3p_1}{(2\pi)^6 \cdot 4E_1E_2} (2\pi)^3 (2\pi)\delta(E_1 + E_2 - E_{TOT})$$

$d^3p_1 = p_1^2 \sin\theta_1\, dp_1 d\theta_1 d\phi_1 = p_1^2 dp_1 d\Omega_1$

$$= \frac{p_1^2\, dp_1\, d\Omega_1}{(2\pi)^3 \cdot 4E_1E_2} (2\pi)\,\delta(E_1 + E_2 - E_{TOT})$$

$$\delta[f(x)] = \sum_{x_0 \in \{\text{zeros of } f\}} \frac{\delta(x - x_0)}{|f'(x_0)|}$$

Convert $\delta$-fct of $E$ to $\delta$-fct of $p_1$ so we can take out the integral. To change variables from $E_1 \to p_1$, we must include a factor of $\left|\frac{\partial(E_1+E_2)}{\partial p_1}\right|^{-1}$.

$$\frac{\partial E_1}{\partial p_1} = \frac{p_1}{E_1} \; ; \; \frac{\partial E_2}{\partial p_1} = \frac{p_1}{E_2}.$$

$$\left|\frac{\partial(E_1+E_2)}{\partial p_1}\right|^{-1} = \left|\frac{p_1}{E_1} + \frac{p_1}{E_2}\right|^{-1}$$

$$= \left|\frac{p_1(E_1+E_2)}{E_1E_2}\right|^{-1}$$

$E_{TOT} = E_1 + E_2.$

$$= \frac{E_1E_2}{p_1 E_{TOT}}$$

Stat. factor — account for degeneracy.

phase factor

This implies

$$\boxed{D = \frac{S}{16\pi^2} \frac{p_1}{E_{TOT}} d\Omega_1}$$

The statistical factor: here, we assume that final state particles are distinguishable - (particle 1 $\neq$ particle 2).

IF multiple identical particles are created, we must include a statistical factor for each.

If $n_i$ particles of type $i$ form post-scatter, then we include the coefficient

$$S_i(n_i) = \frac{1}{n_i!}.$$

D for 3-body final states:

Derivation is straightforward and is the same as that of the 2-body state.

We will find that

$$D = \frac{1}{256\pi^5} dE_1\, dE_2\, d\Omega_1\, d\varphi_{12}$$

where the outgoing particle energies are $E_i$ $(i=1,2,3)$, and we choose the independent variables (for integration) to be $E_1, E_2, \theta_1, \varphi_1$, and $\varphi_{12} \equiv$ the angle between particles 1 and 2 post-scatter.

* In some cases, the amplitude $|A_{fi}|^2$ is independent of $\Omega_1$ and $\varphi_{12}$, which will significantly reduce the complexity of the problem.